**Indian Statistical Institute, Bangalore Centre**

M.Math. I Year, Second Semester

Semestral Examination

Functional Analysis

Time: 3 hours  Date:26 April, 2010  Instructor: T.S.S.R.K.Rao

Maximum Marks 60

Answer all the questions. You can state correctly and use any result proved in class. However if an answer is an immediate consequence of a result quoted, that result also need to be proved. Each question is worth 6 points.

1. Let $c_0 = \{\{\alpha_n\}_{n\geq 1} : \alpha_n \to 0\}$, equipped with the supremum norm. Let $f : c_0 \to \mathbb{C}$ be a linear function such that $f(e_n) = n$. Show that $ker(f)$ is dense in $c_0$.
2. Let $H$ be a complex Hilbert space and let $\Phi : H \to H$ be a linear onto map such that $\|\Phi(x)\| = \|x\|$ for all $x \in H$. Show that $\Phi$ preserves the inner product.
3. Let $A$ be a commutative complex Banach algebra with identity $e$. Show that $A$ is homomorphic and isometric to a subalgebra of the space of bounded linear operators on a Banach space.
4. Let $M = \{f \in C([0,1]) : f(E) = 0\}$, for some closed set $E \subset [0,1]$. Show that the quotient algebra $C([0,1])/M$ is homomorphic and isometric to $C(E)$ .
5. Consider the Hilbert space $\ell^2 = \{\{\alpha_n\}_{n\geq 1} : \sum |\alpha_n|^2 < \infty\}$. Let $T : \ell^2 \to \ell^2$ be defined by $T(\{\alpha_n\}_{n\geq 1}) = \{\frac{1}{n}\alpha_n\}_{n\geq 1}$. Compute $\|T\|$, also describe $T^*$.
6. Let $A$ be a commutative Banach algebra with identity $e$. Show that every proper ideal is contained in a maximal ideal.
7. Let $X$ be a normed linear space and $Y$ a Banach space. Show that the space of bounded linear operators $\mathcal{L}(X,Y)$ is a Banach space.
8. Let $T : L^1([0,1]) \to L^2([0,1])$ be a bounded linear map. Consider $T^* : L^2([0,1]) \to L^\infty([0,1])$ defined by $T^*(f)(g) = \int T(g)\overline{f}\ dx$ for $f \in L^2([0,1])$ and $g \in L^1([0,1]$. Show that $T^*$ is a well-defined, bounded linear map.
9. Let $H$ be a complex Hilbert space. Show that for some infinite discrete set $\Gamma$, there is a bounded linear map from $H$ onto $\ell^2(\Gamma)$.
10. Let $H$ be a complex Hilbert space and let $T : H \to H$ be a bounded linear map such that its adjoint $T^* = 0$. Show that $T = 0$.